数理解析研究所講究録
第 1982 巻 2016 年 8-12

On uniformly-type Sakaguchi functions

Junichi Nishiwaki and Shigeyoshi Owa

Abstract

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disc $\mathbb{U}$. Furthermore, let $\mathcal{US}_s$ and $\mathcal{US}_s(\alpha,\beta)$ be the subclasses of $\mathcal{A}$ consisting of functions $f(z)$ related to uniformly convex and Sakaguchi functions. The object of the present paper is trying to guess inclusive relations between uniformly convexity, $\mathcal{S}_p$ and $\mathcal{US}_s$, and considering coefficient inequalities for $f(z)$ belonging to the class $\mathcal{US}_s(\alpha,\beta)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)\in\mathcal{A}$ is said to be starlike with respect to symmetrical points in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right)>0 \qquad (z\in\mathbb{U}).$$

This class is introduced by Sakaguchi [7]. A function $f(z)\in\mathcal{A}$ is said to be in the class of uniformly convex (or starlike) functions denoted by $\mathcal{UCV}$ (or $\mathcal{UST}$) if $f(z)$ is convex (or starlike) in $\mathbb{U}$ and maps every circle or circular arc in $\mathbb{U}$ with center at ζ in $\mathbb{U}$ onto the convex arc (or the starlike arc with respect to $f(\zeta)$). These classes are introduced by Goodman [1] (see also [2]). For the class $\mathcal{UCV}$, it is defined as the one variable characterization by Rønning [5] and [6], that is, a function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{UCV}$ if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z\in\mathbb{U}).$$

It is independently studied by Ma and Minda [3]. Further, a function $f(z)\in\mathcal{A}$ is said to be the corresponding class denoted by $\mathcal{S}_p$ if it satisfies

2010 Mathematics Subject Classification: Primary 30C45

Keywords and Phrases: Analytic function, Sakaguchi function, uniformly starlike, uniformly convex, parabolic starlike.

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{U}).$$

This class $\mathcal{S}_p$ was introduced by Rønning [5]. We easily know that the relation $f(z) \in \mathcal{UCV}$ if and only if $zf'(z) \in \mathcal{S}_p$. By virtue of these classes, we define the subclass $\mathcal{US}_s(\alpha, \beta)$ of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > \alpha\left|\frac{zf'(z)}{f(z) - f(-z)} - \frac{1}{2}\right| + \beta \qquad (z \in \mathbb{U})$$

for some α ($\alpha \geqq 0$) and β ($0 \leqq \beta < \frac{1}{2}$). We denote $\mathcal{US}_s(1, 0) \equiv \mathcal{US}_s$.

2 Some examples of relation between the class $\mathcal{S}_p$, $\mathcal{UCV}$ and $\mathcal{US}_s$

We don't have the inclusion relation between the class $\mathcal{S}_p$, $\mathcal{UCV}$ and $\mathcal{US}_s$. However, we give two examples to consider some relations between these classes.

Example 2.1. *Let us consider the function $f(z) \in \mathcal{A}$ as given by*

$$f(z) = z + \frac{1}{5}z^3.$$

Then we obtain

$$\begin{aligned}
&\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) - \left|\frac{zf'(z)}{f(z) - f(-z)} - \frac{1}{2}\right| \qquad (z = re^{i\theta}) \\
&= \frac{25 + 20r^2\cos 2\theta + 3r^4}{50 + 20r^2\cos 2\theta + 2r^4} - \frac{r^2}{\sqrt{25 + 10r^2\cos 2\theta + r^4}} \\
&\geqq \frac{(1-r)(5+2r)}{2(5-2r)} > 0
\end{aligned}$$

which shows that $f(z) \in \mathcal{US}_s$. On the other hand, choosing $z = \frac{2}{3}e^{\frac{\pi}{2}i}$, we get

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \left|\frac{zf''(z)}{f'(z)}\right| = -\frac{5}{11}$$

which shows that $f(z) \notin \mathcal{UCV}$.

Example 2.2. *Let us consider the function $f(z) \in \mathcal{A}$ as given by*

$$f(z) = z + \frac{1}{7}z^4.$$

Then we obtain

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) - \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z = re^{i\theta})$$

$$= \frac{49 + 35r^3 \cos 2\theta + 4r^6}{49 + 14r^3 \cos 2\theta + r^6} - \frac{3r^3}{\sqrt{49 + 14r^3 \cos 2\theta + r^6}}$$
$$\geqq \frac{7(1 - r^3)}{7 - r^3} > 0$$

which shows that $f(z) \in \mathcal{S}_p$. *On the other hand, choosing* $z = \frac{23}{24} e^{\frac{\pi}{3} i}$, *we get*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) - \left|\frac{zf'(z)}{f(z) - f(-z)} - \frac{1}{2}\right| = -\frac{71}{24192}$$

which shows that $f(z) \notin \mathcal{US}_s$.

3 Coefficient inequalities for the class $\mathcal{US}_s(\alpha, \beta)$

Our aim of this section is to discuss some coefficient inequalities for function $f(z)$ to be in the class $\mathcal{US}_s(\alpha, \beta)$.

Theorem 3.1. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty} [2(n-1)(1+\alpha)|a_{2n-2}| + \{2n - 1 - 2\beta + 2\alpha(n-1)\}|a_{2n-1}|] \leqq 1 - 2\beta \tag{3.1}$$

for some α $(\alpha \geqq 0)$ *and* β $(0 \leqq \beta < \frac{1}{2})$, *then* $f(z) \in \mathcal{US}_s(\alpha, \beta)$.

Let us consider an example for Theorem 3.1.

Example 3.1. *supposing that the function* $f(z) \in \mathcal{A}$ *as given by*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-2\beta)t\delta_{2n-2}}{2n(n-1)^2(1+\alpha)} z^{2n-2} + \sum_{n=2}^{\infty} \frac{(1-2\beta)(1-t)\delta_{2n-1}}{\{2n - 1 - 2\beta + 2\alpha(n-1)\}n(n-1)} z^{2n-1}$$

for some α $(\alpha \geqq 0)$, β $(0 \leqq \beta < \frac{1}{2})$, t $(0 \leqq t \leqq 1)$ *and* $|\delta_{2n-2}| = |\delta_{2n-1}| = 1$. *Then the coefficient* (3.1) *yields*

$$\sum_{n=2}^{\infty} [2(n-1)(1+\alpha)|a_{2n-2}| + \{2n - 1 - 2\beta + 2\alpha(n-1)\}|a_{2n-1}|]$$
$$= \sum_{n=2}^{\infty} \left\{\frac{(1-2\beta)t}{n(n-1)} + \frac{(1-2\beta)(1-t)}{n(n-1)}\right\}$$
$$\leqq 1 - 2\beta.$$

This implies that $f(z) \in \mathcal{US}_s(\alpha, \beta)$.

Theorem 3.2. *If $f(z) \in \mathcal{US}_s(\alpha, \beta)$, then*

$$|a_2| \leqq \frac{1-2\beta}{|1-\alpha|},$$

$$|a_3| \leqq \frac{1-2\beta}{|1-\alpha|},$$

$$|a_{2n}| \leqq \frac{1-2\beta}{n|1-\alpha|} \prod_{j=1}^{n-1} \left(1 + \frac{1-2\beta}{j|1-\alpha|}\right) \qquad (n = 2, 3, 4, \cdots)$$

and

$$|a_{2n+1}| \leqq \frac{1-2\beta}{n|1-\alpha|} \prod_{j=1}^{n-1} \left(1 + \frac{1-2\beta}{j|1-\alpha|}\right) \qquad (n = 2, 3, 4, \cdots).$$

References

[1] A. W. Goodman, *On uniformly convex functions*, Annal. Polon. Math. **56**(1991), 87 - 92.

[2] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155**(1991), 364 - 370.

[3] W. Ma and D. Minda, *Uniformly convex functions*, Annal. Polon. Math. **57**(1992), 165 - 175.

[4] S. Owa, T. Sekine, R. Yamakawa, *Notes on Sakaguchi functions*, Austral. J. Math. Appl. **3** (12) (2005), 1 - 7.

[5] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118**(1993), 189 - 196.

[6] F. Rønning, *On uniform starlikeness and related properties of univalent functions*, Complex Variables **24**(1994), 233 - 239.

[7] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11**(1959), 72 - 75.

[8] S. Shams, S. R. Kulkarni, and J. M. Jahangiri, *Classes of uniformly starlike and convex functions*, Internat. J. Math. Math. Sci. **55**(2004), 2959 - 2961.

Junichi Nishiwaki
Department of Mathematics and Physics
Setsunan University
Neyagawa, Osaka 572-8508 Japan
email:*jerjun2002@yahoo.co.jp*

Shigeyoshi Owa
Department of Mathematics
Faculty of Education
Yamato University Katayama 2-5-1, Suita, Osaka 564-0082
Japan
email: *owa.shigeyoshi@yamato-u.ac.jp*